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THE THEORY OF GRAVITATION IN SPECIAL RELATIVITY†

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It is shown how, based on the history of the development of the theory of gravitation, one can state and effectively solve problems in the mechanics of the motion of a system of individually defined bodies, not in contact with one another, but interacting with mass forces. The theory is constructed using a model system of singular points which possess (in non-holonomically defined comoving Fermi coordinates) proper time, masses, thermodynamic and potential energies and constant angular momenta, and are embedded in Minkowski four-space. The appropriate comoving metrics and equations of motion for these points in Fermi coordinates are written down directly. To obtain the laws of motion from the standpoint of given observers one must use algorithms of calculations in the theory of inertial navigation in Riemannian spaces, as developed and published by the author.

In special relativity theory (SRT) one postulates a globally four-dimensional Minkowski space, while in general relativity theory (GRT) one essentially postulates the validity of the metric of SRT only along the coordinate lines L of proper time τ for individual points with elementary masses dm. Concerning these points, one postulates in the theory of "pure gravitational phenomena" that dm = const and that changes in dm owing to radiation or due to adhesion or separation of bodies are not taken into account during the motion; the interaction of different elements dm owing to contacts (internal stresses and, in particular, pressures) is also not taken into account. One can formulate problems in which all the interactions just listed, as well as interactions of other types, are modelled and taken into consideration in GRT. However, a (Newtonian) theory of gravitation in both GRT and SRT can be constructed taking into account only interactions of moving individual point masses and the geometric properties of four-dimensional continua of geometrical points with coordinates $\xi^1, \xi^2, \xi^a, \tau$ of pseudo-Riemannian spaces with signature $---+$, for which the distances ds between any infinitesimally close points may be defined by a metric form of the type [1]

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma, \tau) d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\gamma, \tau) d\xi^\alpha d\xi^\beta \tag{1}$$

$$\gamma, \alpha, \beta = 1, 2, 3$$

where c is a fixed empirical scalar constant, given as a characteristic of the pseudo-Riemannian space, the numbers ξ^1, ξ^2, ξ^3 are the coordinates (names) of individual points, and the time coordinate τ is a model proper time on the trajectory $L(\xi^\gamma)$, measured in hours, of individual points fastened with model coordinates ξ^α = const (L) for $d\xi^\alpha = 0$ on L.

For a fixed space, the metric in (1) is not uniquely defined. It is easy to see that under any transformation of the following type [2]

$$\eta^\alpha = \varphi^\alpha(\xi^\gamma) \text{ and } \tau' = \tau + \Omega(\xi^\gamma) \text{ when } d\tau' = d\tau \text{ on } L \tag{2}$$

the metric (1) preserves its form and that all invariant characteristics of the space geometry retain their values. In particular, the coordinate lines L are preserved with invariant values of the variables τ, except for displacement along L of the origin of the proper time on lines L.

At each point along lines L one can introduce an invariantly defined vector of absolute four-dimensional velocity $\mathbf{u}$ and acceleration $\mathbf{a}$ as derivatives with respect to the variable τ along trajectories of individual points with (ξ^γ) = const(L)

$$\mathbf{u} = d\mathbf{s}/d\tau = \mathbf{c}, \quad \mathbf{a} = d\mathbf{u}/d\tau \tag{3}$$

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The vector **u** always points along the tangent to the corresponding four-dimensional trajectory L.

In the metric (1) for individual points, the acceleration vector **a** along a line L is always perpendicular to the four-dimensional velocity **u**, since $|\mathbf{u}| = c$ is a constant.

With every point of a line L one can associate a local tetrad reference system with vectors $\ni_k$, in which the basis vector $\ni_4 = \mathbf{u}$ points along the tangent to L and, since

$$\ni_4 = g_{p4}\ni^p, \quad \mathbf{a} = \frac{d\mathbf{u}}{d\tau} = c\frac{\partial g_{p4}}{\partial \tau}\ni^p + \frac{c}{2}g_{k4}\Gamma^k_{p4}\ni^p =$$

$$= c\frac{\partial g_{p4}}{\partial \tau}\ni^p + \frac{c}{2}\delta^s_4\left(\frac{\partial g_{sp}}{\partial \tau} + \frac{\partial g_{s4}}{\partial \xi^p} - \frac{\partial g_{p4}}{\partial \xi^s}\right) = c\frac{\partial g_{\alpha 4}}{\partial \tau}\ni^\alpha$$

because $u_p = cg_{\rho 4}$, $\mathbf{u} = u_p\ni^p$, it follows that the acceleration in the comoving coordinate system is given by [2]

$$a_\alpha = c\partial g_{\alpha 4}(\xi^\alpha, \tau)/\partial \tau, \quad a_4 = 0 \tag{4}$$

It is obvious that the components of the metric $g_{\alpha\beta}(\xi^\alpha, \tau)$ do not affect the magnitude of the vector **a**, since, under transformations (2), the components of the acceleration in variables η^α, τ' are defined by the following formulae [2]

$$a_\alpha = a_s\partial \xi^s/\partial \eta^\alpha, \quad a_4 = 0 \tag{5}$$

On different coordinate lines L, non-holonomic transformations can be used to introduce Fermi coordinates x^α, τ in which formulae (4) will be valid after transformations, with the sole difference that the variables ξ^α will be replaced b y x^α. These coordinates may be considered as different corresponding Cartesian coordinates with corresponding different constant values on the lines L, and the transformed components $g'_{\alpha\beta}(x^\gamma, \tau)$ are also represented non-holonomically in terms of them.

Thus, for each value of the coordinate τ and each corresponding law of motion defined by the values of x^γ, τ' in the components $g'_{\alpha 4}(x^\gamma, \tau')$ and in the components of metrics $g'_{\alpha\beta}(x^\gamma, \tau')$ obtained on the lines L, one can consider a locally defined and essentially global corresponding pseudo-Riemannian space.

The model construction just described, with different values of the constant x^γ on different laws of motion, conforming to coordinate lines L for global time τ and corresponding accelerations **a**, can be supplemented by fixed components of the three-dimensional metric $g'_{\alpha\beta}(x^\gamma, \tau)$. In this way one can single out non-holonomic corresponding pseudo-Riemannian spaces S with given acceleration fields of gravitational forces in a system of Fermi coordinates x^γ, τ on a distinguished, generally arbitrary family of lines L for the scalar global variable τ.

To specify the acceleration field, one must rely on the formulation of model conditions corresponding to observations and experiments, which generate the form of the family of lines L and of the corresponding pseudo-Riemannian space.

To determine the coordinate lines L, one should use model mechanisms of the realization of the weightlessness observed in the theory of gravitation in motions of all the small individual masses.

Weightlessness of moving bodies is due to the balance of forces proportional to the masses at all the individual points of the bodies. In the theory of gravitation these are model attractive forces, forces balanced by inertial forces, or simply inertial motions when there are no interactions between the various sets of points made up of the moving individuals.

The basic characteristic examples of balances are associated with the universal local equation of mechanics for motion of a small body in vacuum

$$\mathbf{P} - m\mathbf{a} = 0 \tag{6}$$

where **P** is a quantity proportional to the mass of the small body, m is the mass and **a** is the acceleration relative to the inertial tetrad $dx^1dx^2dx^3$, $d\tau$.

Equations (6) are locally universal both in the Newtonian theory of motion in three-dimensional Euclidean space using synchronized absolute time, and in curved Riemannian spaces, in which local relations, in particular Eq. (6), are postulated for identical **P**, **a** and m, which is possible and natural in Riemannian spaces.

In Eq. (6) the balance of forces may be treated in Newton's sense, as an equilibrium under the action of two non-zero forces: a gravitational force $\mathbf{P} \neq 0$ and an inertial force $-m\mathbf{a} \neq 0$, and the balance of these forces at all individual points yield weightlessness. (The explanation of the phenomenon of weightlessness for $\mathbf{P} \neq 0$ and $\mathbf{a} \neq 0$ in the Newtonian theory may be confirmed in experiments using inertial instruments mounted on moving bodies.)

In GRT weightlessness is treated as the absence of a gravitational force ($\mathbf{P} = 0$) for points freely moving in a gravitational field, and there should therefore be no acceleration: $\mathbf{a} = 0$. Thus, weightlessness is the absence of a gravitational force in free motion (provided the gravitational theory involves no other external forces or internal stresses). In GRT all the individual masses cause the acceleration $\mathbf{a}$ to vanish along each line L, while there is no conservation of the gravitational force ($\mathbf{P} \neq 0$), and this yields weightlessness in GRT.

In GRT all Ls are "straight lines" and the motion takes place along geodesics; but in the Newtonian theory when motion takes place weight is balanced by the inertial force, as a result of which the phenomenon of weightlessness occurs in the Newtonian theory also.

Using formulae (2) and (4) in GRT in the metric (1) we obtain

$$\partial g_{\alpha 4}/\partial \tau = 0 \tag{7}$$

and consequently $g_{\alpha 4} = g_{\alpha 4}(\xi^{\alpha})$ and formula (1) becomes, in the comoving global system of coordinates

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^{\gamma})d\xi^{\alpha}d\tau + g_{\alpha\beta}(\xi^{\gamma}, \tau)d\xi^{\alpha}d\xi^{\beta} \tag{8}$$

If the acceleration vectors $\mathbf{a}$ according to (4) are not zero, one can write for the corresponding trajectories, for arbitrary $d\xi^{\alpha}$

$$\frac{\partial}{\partial \tau}(2g_{\alpha 4}(\xi^{\gamma}, \tau')d\xi^{\gamma}d\tau = \frac{2}{c}a_{\alpha}d\xi^{\alpha}d\tau = -\frac{2}{c}dUd\tau \neq 0 \tag{9}$$

If ξ^1, ξ^2, ξ^3 are regarded as coordinates of individual points with masses dm, then, in accordance with the transformation (2), one can write down a local formula for the elementary work performed in variational or real displacements of individual points [2]

$$dma_{\alpha}d\xi^{\alpha} = dma'_{\alpha}d\eta^{\alpha} = dma_{\alpha}dx^{\alpha} = -dmdU \tag{10}$$

where $dU = -a_{\alpha}dx^{\alpha}$ in Fermi variables.

The right-hand side of (10) may be regarded as the increment of elementary energy $dmdU$ due to an increment in the coordinates $d\xi^{\alpha}$ or $d\eta^{\alpha}$ or dx^{α}. Accordingly, the scalar quantity dU may be considered as the increment of energy density.

In the Newtonian theory of gravitation the infinitesimal scalar dU is the increment of the potential energy density, which is uniquely defined by a function $mU(x^{\gamma})$ which depends only on the Fermi coordinates x^{γ}, while Eq. (10) serves as the definition of the total differential for $dU(x^{\gamma})$.

It has been shown [3] that, in terms of Fermi variables, the components of the absolute acceleration a_{α} kinematically in the Newtonian sense and in SRT in comoving coordinates are identical. This obviously follows from the fact that they are the same in comoving coordinates at the points of arbitrary coordinate lines of τ, which are denoted by L.

This situation obviously remains valid locally and in pseudo-Riemannian spaces in comoving coordinates, and therefore the small quantity dU is a total differential in SRT and in GRT.†

Hence it follows that in comoving coordinates the following fundamental kinematic equations of celestial mechanics must hold in Fermi variables

$$\partial U / d\tau = 0, \quad ma_{\alpha}(x^{\gamma}) = -m\partial U(x^{\gamma}) / \partial x^{\alpha} \tag{11}$$

†The fact that the function U is constant along L is proved by kinematic arguments, as follows. The four-dimensional velocity $\mathbf{u}$ points along the tangent to L, the vector $\mathbf{a}$ points along the normal, since along L the acceleration $\mathbf{a}$ and ds on L are perpendicular and consequently $\mathbf{a}ds = 0 = -d'U$ on L. Hence, the uniqueness of the scalar functions $U(\xi^1, \xi^2, \xi^3)$ or $U(x^1x^2x^3)$, defined locally on L in non-holonomically introduced Fermi coordinates, implies the existence of a scalar potential for the acceleration vector $\mathbf{a}$.

Equations (11) may also be derived independently in local inertial frames of reference as dynamical equations [4], if one stipulates that in the energy balance equations of the individual masses a potential energy appears that depends only on the coordinates x^α or ξ^α.

One should also remember that the potential energy depends only on the coordinates of the individual points of a system of moving masses for conservative systems in comoving coordinates x^γ or ξ^γ, when the relative velocity is zero, as follows directly from the energy equation.

Relationships (4) and Eqs (11) yield

$$a_\alpha = c\partial g_{\alpha 4} / \partial\tau = -\partial U(x^\gamma) / \partial x^\alpha \neq 0$$

Hence, allowing for the additive sum of tensor terms of the components with $g_{\alpha 4}(\xi^\gamma)$ along each coordinate line L, one can write

$$g_{\alpha 4} = -\frac{1}{c}\tau\frac{\partial U}{\partial x^\alpha} + g_{\alpha 4}(\xi^\tau) \tag{12}$$

and instead of formula (9), we deduce from (12) that on L

$$ds^2 = c^2 d\tau^2 - 2\frac{\tau}{c}dU d\tau + 2g_{\alpha 4}(\xi^\gamma)d\xi^\alpha d\tau + g_{\alpha\beta}dx^\alpha dx^\beta \tag{13}$$

But now, if $a_\alpha \neq 0$ along each line L—the coordinate line for global time τ, it is true that $dU = 0$, and therefore

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma)d\xi^\alpha d\tau + g_{\alpha\beta}(x^\gamma, \tau)dx^\alpha dx^\beta \tag{14}$$

The global forms of the function $g_{\alpha\beta}(x^\gamma, \tau)$ in constructions of the metric may be assumed to be the same for some L or different for different L. The space metrics (9) in GRT and (14) in the general case cannot be transformed to synchronous form. However, given a function $U(x^\gamma)$, which is different for different individual points, the motion will take place according to laws defined by Eqs (11). It is very important to emphasize that all the relationships obtained above are mathematically valid for arbitrary given functions $U(x^\gamma)$. Therefore, in order to obtain specific solutions, one must still specify equations to define the potential energy density $U(L) = U(x^\gamma)$.

In Newtonian mechanics, one uses for $U(x^\gamma)$ the law of universal gravitation, which has been verified by direct experiments, by a great many results of observations of the motion of masses in celestial mechanics and, in particular, in cases of large four-dimensional accelerations of motion of individual objects.

In a theory of "pure gravitation" it is not possible to obtain physical intersections or tangencies of different L lines, when one must construct complicated models taking into account different contact interactions of individuals.

The arbitrariness of the function $U(x^\gamma)$ in the metric (13) indicates that the equations of GRT are not closed, as they are derived without explicit allowance for the law of universal gravitation with the postulated hopes that it should be possible to replace accelerations in Minkowski space by curvatures of pseudo-Riemann spaces.

Some attempts have been made to prove the assumption that effects of curved pseudo-spaces are automatically compatible as a substitute for accelerated motions in Newtonian mechanics; these attempts cannot be considered to be correct [2].

It is easily shown that, if rotation occurs relative to the inertial tetrads of individual particles (in the general case, when the velocity fields are rotational), when rot $\mathbf{u}(\xi^\gamma) = 2\boldsymbol{\omega} = \text{const} \neq 0$ on L, which is due to the presence of a constant angular momentum (e.g. for rotating stars or planets), it is impossible to reduce the transformations of comoving metrics (9) and (14) to a synchronous form.

It is obvious that if $\boldsymbol{\omega}(\xi^\gamma) \neq 0$ because of rotation in orbits of individual bodies, additional four-dimensional accelerations of individual objects will arise owing to the presence of constant angular momenta.

There are obviously examples of the motion of stellar and planetary systems in which the geodesic property of the orbits may be substantially violated because the stars and planets possess large angular momenta and because there is an additional potential energy density in the expression for the function $U(x^\gamma)$.

To obtain data about the metric and about the laws of motion of individuals for given observers on the basis of the theory presented above, it is necessary to derive a preliminary solution in comoving frames of reference and then to recalculate the solutions, using an algorithm of inertial navigation in pseudo-Riemannian spaces [4].

In view of the theory developed here, the universality of the use of Minkowski space in present-day physical theories (as is actually employed in practical applications in physics and engineering), like Newtonian theory, is obviously natural and desirable for applications in many physical models, as is actually corroborated in actual applications.

APPENDIX

To clarify the essence of a model theory of the mechanics of gravitation in relativity theory it will be useful to note the following relationships.

In any four-dimensional pseudo-Riemannian space one can introduce comoving Lagrange coordinates and coordinate lines L for an invariant global time variable with $\tau \neq 0$ on the L lines, and coordinates of individual points ξ^1, ξ^2, ξ^3 with a non-uniquely defined comoving metric of the form

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma, \tau) d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\gamma, \tau) d\xi^\alpha d\xi^\beta \qquad \text{(A1)}$$

As is well known, the field equations in the theory of gravitation are

$$R_i^j - \frac{1}{2}\delta_i^j R = k\rho u_i u^j \qquad \text{(A2)}$$

with a constant $k = 8\pi G c^{-4} = 2.07 \times 10^{-48}$ $s^2/(g\ cm)$, where the vector field of the four-dimensional velocity

$$\mathbf{u} = u_k \mathfrak{Э}^k = u_p \mathfrak{Э}^p \quad \text{for } |\mathbf{u}| = c$$

points along the tangent at points of the coordinate lines L.

Attention should be directed to the validity of the following equations for any m and k

$$g_{k4}\mathfrak{Э}^k\mathfrak{Э}^4 = T \text{ and } \nabla_m T = \nabla_4 T = 0 \text{ (i.e. } \nabla_4 g_{k4} = 0)$$

$$c g_{k4}\mathfrak{Э}^k = \bar{u} \text{ and } \nabla_4 c g_{k4} = \nabla_4 u_k = a_k, \text{ generally speaking, for } a_k \neq 0.$$

Now, reasoning in GRT, it follows from (A2) and from the Bianchi conditions that always

$$\nabla_j \left(R_i^j - \frac{1}{2}\delta_i^j R \right) = 0$$

and therefore it follows from equations (A2) on the L lines holonomically or in Fermi variables non-holonomically that

$$\nabla_j \rho u^j u_i = u_i \nabla_j \rho u^j + \rho u^j \nabla_j u_i = 0, \quad u^4 = c, \quad u^\alpha = 0$$

The law of conservation of mass for individuals gives $\nabla_j(\rho u^j) = 0$, so that on coordinate lines L, for variables r, one obtains $\nabla_{4u_i} = 0$ or $\mathbf{u} =$ const, and, consequently, free motion of all individual points is inertial, the L lines are geodesics and therefore *the force of gravity is generally absent*. This is the main conclusion as to weightlessness in the theory of gravitation of GRT.

In Newtonian mechanics one has a potential energy, and weightlessness follows from the balance of the gravitational force and the inertial force on L, holonomically in variables ξ^α, τ or non-holonomically in Fermi variables x^γ, τ

$$\mathbf{P} - m\mathbf{g} = 0$$

where $\mathbf{g} = \mathbf{a}$ is the acceleration due to gravity, which is absolute or relative (e.g. for astronauts) when the relative velocity is zero.

Next, after suitable contractions and conditions imposed on the comoving coordinates, using (A2), it follows that

$$-c^2 R_4^4 = c^2 R/2 = -4\pi\rho G \qquad \text{(A3)}$$

Independently of Eqs (A2) and their corollaries (A3), the energy equation in the theory of gravitation must involve not only the energy mc^2 but also a potential energy scalar mU.

Regardless of the validity of the usual inequality $U \ll c^2$, both in natural phenomena and in engineering problems, the potential energy is of fundamental importance, due to which it is possible to construct celestial mechanics for the Newtonian theory of gravitation and in Minkowski space in SRT.

We know that if a scalar density is introduced for the potential energy in comoving coordinates on each L line, then $U(\xi^1, \xi^2, \xi^3) = U(x^1x^2x^3)$ for individual bodies in the theory of gravitation.

On the basis of a great many experiments, the function U may be specified in terms of finite formulae or equivalently by means of Poisson's equation

$$\Delta U = -4\pi\rho G \tag{A4}$$

In vacuous volumes, with $\rho = 0$, the function U for separate moving test particles is harmonic.

Equation (A4) must be considered as an addition to the system of non-closed equations (A2).

For spaces that are topologically equivalent to Minkowski space with Fermi coordinates, the solutions of the scalar Poisson's equations (A4) for scalars $U(x^\gamma)$ with given $\rho(x^\gamma)$ are independent of the metric. A suitable function $U(x^\gamma)$ in the comoving metric [1] may be chosen arbitrarily, since $dU = 0$ on every L line. It is obvious that a pseudo-Riemannian space in GRT is defined by any metric

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha\xi}(\xi^\gamma) d\xi^\alpha d\tau + g_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{A5}$$

with embedded terms that depend on $U(x^\gamma)$.

It is obvious that only in vacuous volumes, for which $R_1^4 = 0$, $R = 0$ and $\Delta U = 0$, can one pose the question of whether equalities (A3) and (A4) are compatible. In that case too, however, it follows from (A3) that the orbits are geodesics, while according to (A4) and (A5) accelerated motions appear in the corresponding orbits. Therefore, for every fixed solution in the Newtonian theory of gravitation or in SRT, it is impossible to consider corresponding orbits as mathematical limits of corresponding geodesic lines L; this is particularly important for large intervals of proper time τ.

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